Weighted voting systems: reliability versus rapidity

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Abstract

The weighted voting system (WVS) consists of n units that each provide a binary decision (0 or 1) or abstain from voting. Each unit has its own individual weight. System output is 1 if the cumulative weight of all 1-opting units is at least a pre-specified fraction τ of the cumulative weight of all non-abstaining units. Otherwise, system output is 0. The system input is either 0 or 1. Every unit is characterized by probability of making decisions 0 and 1 and by probability of abstaining for each input. The system fails if its output is not equal to its input.

This paper shows that if the WVS consists of units that need different time to produce their outputs, the decision time of the entire system depends on the distribution of unit weights and on the value of τ. It shows also that a tradeoff exists between the system reliability and its rapidity.

An algorithm that finds the system parameters maximizing its reliability under constraint imposed on the expected system decision time is suggested. Illustrative examples are presented.

Notation

τ	threshold factor, $(0 \le \tau \le 1)$
n	number of units belonging to WVS
K	number of different voting results in WVS
I	WVS input (proposition to be accepted or rejected), $I \in \{0,1\}$
$\Pr\{v\}$	probability of event v
p_i	$\Pr\{I = i\}$
$d_j(I)$	decision (output) of individual unit j

$D(I)$ decision of the entire WVS

$q_{is}^{(j)}$ $\Pr\{d_j(I) = s \mid I = i\}$, $i \in \{0,1\}$, $s \in \{0,1,x\}$

Q_{is} $\Pr\{D(I) = s \mid I = i\}$, $i \in \{0,1\}$, $s \in \{0,1,x\}$

R $\Pr\{D(I) = I\}$, WVS reliability

$U_I^m(z)$ U-function representing all the possible different states of a subsystem containing m fastest voting units when system input is I

w^0_j "negative" weight of unit j (weight of unit j when $d_j(I)=0$)

w^1_j "positive" weight of unit j (weight of unit j when $d_j(I)=1$)

t_j time when the decision $d_j(I)$ of voting unit j becomes available

T expected WVS decision time

T^* maximum allowed expected decision time

W_m^1 total weight of units that have decision time not greater than t_m and vote for the proposition acceptance

W_m^0 total weight of units that have decision time not greater than t_m and vote for the proposition rejection

h_{I0}^m probability that the proposition I is rejected at time t_m

h_{I1}^m probability that the proposition I is accepted at time t_m

$1(x)$ unity function: 1(TRUE)=1, 1(FALSE)=0

Acronyms[1]

WVS weighted voting system

UF $U(z)$-function

[1] The singular & plural forms of acronyms are always spelled the same.

1. Introduction

Voting systems (k-out-of-n systems with multiple failure modes) are widely used in human organization systems as well as in technical decision making systems. The reliability of these systems has been studied in [1-6]. The weighted voting systems (WVS) are generalization of the voting systems. WVS are intensively studied in recent years [7-12]. The applications of WVS can be found in imprecise data handling [13], safety monitoring and self-testing [14], multi-channel signal processing [15], pattern recognition and target detection [7], etc.

A WVS makes a decision about propositions based on the decisions of n statistically independent individual units of which it consists (for example, in target detecting system speed detectors and heat radiation detectors provide the system with their individual decisions without communicating among themselves). Each proposition is *a priori* right or wrong but this information is available for the units in implicit form. Therefore the units are subject to the following three errors:

1. Acceptance of a proposition that should be rejected (fault of being too optimistic),
2. Rejection of a proposition that should be accepted (fault of being too pessimistic),
3. Abstaining from voting (fault of being unavailable or indecisive).

This can be modeled by considering system input I being either 1 (proposition to be accepted) or 0 (proposition to be rejected) which is supplied to each unit. Each unit j produces its decision (unit output) $d_j(I)$ which can be 1, 0 or x (in the case of abstention). Inequality $d_j(I)\neq I$ means that the decision made by the unit is wrong. The listed above errors can be expressed as

1. $d_j(0)=1$ (unit fails stuck-at-1),
2. $d_j(1)=0$ (unit fails stuck-at-0),
3. $d_j(I)=x$ (unit fails stuck-at-x).

Accordingly, reliability of each unit j can be characterized by probabilities of these errors: $q_{01}^{(j)}$ for the first one, $q_{10}^{(j)}$ for the second one, $q_{1x}^{(j)}$ and $q_{0x}^{(j)}$ for the third one (note that stuck-at-x probabilities can be different for inputs I=0 and I=1).

In this paper we consider an asymmetric WVS (first suggested in [9]) which is able to take advantage of knowledge about statistical asymmetry of voting units (asymmetric probabilities of making correct decisions with respect to the input I) and therefore has greater reliability than the symmetric WVS. In such system each voting unit j has two weights that express its relative importance in the WVS: "negative" weight w^0_j which is assigned to the unit when it votes for the proposition rejection and "positive" weight w^1_j which is assigned to the unit when it votes for the proposition acceptance. To make a decision about proposition acceptance, the system incorporates all the unit decisions into a unanimous system output D (Fig. 1). The proposition is rejected by the WVS ($D(I)=0$) if the total weight of units voting for its acceptance is less than a pre-specified fraction τ of total weight of not abstaining units (τ is usually referred to as threshold factor). The WVS abstains ($D(I)=x$) if all of its voting units abstain.

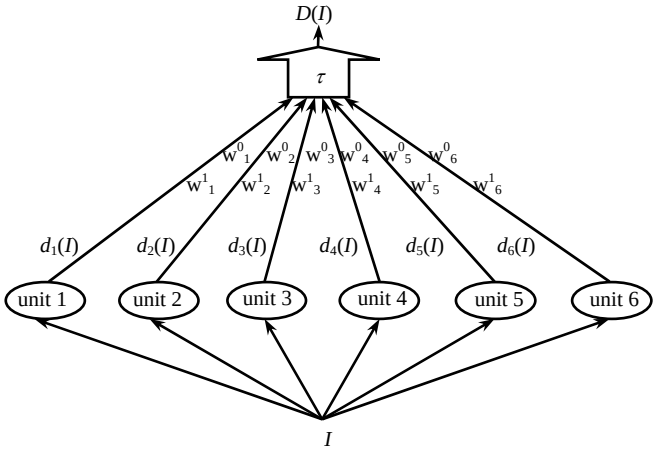


Fig. 1. Example of asymmetric WVS with $n=6$.

The system fails if $D(I)\neq I$. The entire WVS reliability can be defined as $R=\Pr\{D(I)=I\}$. One can see that the system reliability is a function of reliabilities of units it consists of. The reliability characteristics of WVS units as well as the propositions probability distribution can be elicited from historical statistics. In technical systems probabilities of different kinds of errors can be obtained for each unit with high precision by intensive testing. The entire WVS reliability also depends on the unit weights and the threshold. While the units' reliabilities usually can not be changed when the

WVS is built, the weights and the threshold can be chosen in such a way that maximizes the entire WVS reliability:

$$(w^0_1, w^1_1, \ldots, w^0_n, w^1_n, \tau) = \arg\{R(w^0_1, w^1_1, \ldots, w^0_n, w^1_n, \tau) \rightarrow \max\}. \quad (1)$$

The algorithm for WVS optimization according to this formulation is presented in Ref. [9].

The previous studies of WVS don't address the aspect of the system rapidity in making decisions. The first paper that considers the decision time factor is by Xie and Pham [11]. In this paper it is assumed that the probabilities of making mistakes decreases with the time for each voting unit but the cost of decision making by the entire system increases with the time. An algorithm for finding the optimal stopping time (the time when all the units are obliged to vote) is suggested. This model is relevant for human organizations in which the requirement of a limited time to complete the report (to make the decision) affects the results of the experts' evaluations together with their expertise.

In many technical systems the time when the output (decision) of each voting unit is available is predetermined. For example, the decision time of a chemical analyzer is determined by the time of a chemical reaction. The decision time of a target detection radar system is determined by the time of the radio signal return and by the time of signal processing by the electronic subsystem. In both these cases the variation of the decision times is usually negligible small.

On the contrary, the decision time of the entire WVS composed from voting units with different constant decision times can vary. As it was mentioned in [11], the system does not need to wait for decisions of slow voting units, as long, as the system can make a correct decision with reliability higher than a pre-specified level. Moreover, in some cases the decisions of the slow voting units do not affect the decision of the entire system since this decision becomes evident after the fast units have voted. This happens when the total weight of units voting for the proposition acceptance or rejection is enough to guarantee the system decision independently of the decisions of the units that have not voted yet. In such situations the voting process can be terminated without waiting for slow units' decisions and the WVS decision can be made in a shorter time.

The number of combinations of unit decisions that allow the entire system decision to be obtained before the outputs of all of the units become available depends on the unit weights distribution and on the threshold value. By increasing the weights of the fastest units one makes the WVS more decisive in the initial stage of voting and therefore reduces the mean system decision time by the price of making it less reliable.

In applications where the WVS should make many decisions in a limited time the expected system decision time is considered to be a measure of its performance. Since the units' weights and the threshold affect both the WVS's reliability and its expected decision time, the problem of the optimal system turning can be formulated as follows: find the voting units' weights and the threshold that maximize the system reliability R while providing the expected decision time T not greater than a pre-specified value T^*:

$$(w^0_1, w^1_1, \ldots, w^0_n, w^1_n, \tau) = \arg\{R(w^0_1, w^1_1, \ldots, w^0_n, w^1_n, \tau) \to \max \mid T(w^0_1, w^1_1, \ldots, w^0_n, w^1_n, \tau) \le T^*\}. \quad (2)$$

This paper presents an algorithm for solving this optimization problem. Section 2 of the paper presents the voting model. Section 3 describes an algorithm for reliability and performance evaluation. Section 4 is devoted to optimization technique.

2. The weighted voting model

Consider the WVS consisting of n voting units characterized by error probabilities $q_{01}^{(j)}$, $q_{10}^{(j)}$, $q_{1x}^{(j)}$, $q_{0x}^{(j)}$ and decision times t_j. The WVS incorporates all the unit outputs into a unanimous system output D using the following threshold based decision rule:

$$D(I) = \begin{cases} 1, & if \quad \sum\limits_{d_j(I)\neq x} w_j^1 d_j(I) \ge \tau \sum\limits_{d_j(I)\neq x} [w_j^1 d_j(I) + w_j^0 (1-d_j(I))], \quad \sum\limits_{d_j(I)\neq x} w_j \neq 0 \\ 0, & if \quad \sum\limits_{d_j(I)\neq x} w_j^1 d_j(I) < \tau \sum\limits_{d_j(I)\neq x} [w_j^1 d_j(I) + w_j^0 (1-d_j(I))], \quad \sum\limits_{d_j(I)\neq x} w_j \neq 0 \\ x, & if \quad \sum\limits_{d_j(I)\neq x} w_j = 0 \end{cases} \quad (3)$$

Let us order the units such that $t_j \le t_{j+1}$ for $1 \le j \le n$-1 and define the total weight of WVS units that have the decision times not greater than t_m and support the proposition I as W_m^1:

$$W_m^1 = \sum_{j=1}^{m} w_j^1 1(d_j(I) = 1) \tag{4}$$

and the total weight of WVS units that have the decision times not greater than t_m and reject the proposition as $W_m{}^0$:

$$W_m^0 = \sum_{j=1}^{m} w_j^1 1(d_j(I) = 0)\,. \tag{5}$$

The decision rule (3) can now be rewritten as follows:

$$D(I) = \begin{cases} 1, & if \quad W_n^1 \geq \tau(W_n^1 + W_n^0), \quad W_n^1 + W_n^0 \neq 0 \\ 0, & if \quad W_n^1 < \tau(W_n^1 + W_n^0), \quad W_n^1 + W_n^0 \neq 0 \\ x, & if \quad W_n^1 + W_n^0 = 0. \end{cases} \tag{6}$$

Following this expression the condition that $D(I)$=0 can be rewritten as

$$W_n{}^1 < \tau(\, W_n{}^1 + W_n{}^0) \tag{7}$$

or

$$(1-\tau)W_n{}^1 - \tau\, W_n{}^0 < 0. \tag{8}$$

This gives one the simple way of tallying the units' votes: each unit j adds value of $(1-\tau)w^1{}_j$ to the total WVS score if it votes for proposition acceptance, value of $-\tau w^0{}_j$ if it votes for proposition rejection and nothing if it abstains. The proposition is rejected if the total score is negative.

Let $V^1{}_i$ and $V^0{}_i$ be the sum of "positive" and "negative" weights of units from i to n respectively:

$$V_i^1 = \sum_{j=i}^{n} w^1{}_j\,,\; V_i^0 = \sum_{j=i}^{n} w^0{}_j\,, \tag{9}$$

($V_{n+1}^1 = V_{n+1}^0 = 0$ by definition).

After the voting of the first m units the total system score is $(1-\tau)W_m{}^1 - \tau W_m{}^0$. The maximal possible value of the score after the rest of the units add their votes is

$$(1-\tau)W_m{}^1 - \tau\, W_m{}^0 + (1-\tau)V_{m+1}^1 \tag{10}$$

(if all of the units from m+1 to n vote for proposition acceptance) and the minimal possible value of the score is

$$(1-\tau)W_m^1-\tau W_m^0-\tau V_{m+1}^0 \quad (11)$$

(if all of the units from m+1 to n vote for proposition rejection). If the maximal possible score is negative

$$(1-\tau)W_m^1-\tau W_m^0+(1-\tau)V_{m+1}^1<0 \quad (12)$$

the proposition will be rejected independently of the decisions of the units m+1, …, n. Therefore there is no need to continue the voting and the proposition can be rejected by the system at time t_m. If the minimal possible score is not negative

$$(1-\tau)W_m^1-\tau W_m^0-\tau V_{m+1}^0\geq 0 \quad (13)$$

there is no chance that the system will reject the proposition even if the units m+1, …, n vote for its rejection. Therefore the proposition can be accepted by the system at time t_m without waiting for decisions of units m+1, …, n.

3. Evaluation of WVS reliability and expected decision time

3.1. Universal generating function technique

The procedure used for WVS reliability and expected decision time evaluation is based on the universal moment generating function technique. This procedure is a modification of the algorithm presented in [9].

The universal moment generating function of a discrete random variable X is defined as a polynomial

$$U(z)=\sum_{k=1}^{K} s_k z^{x_k} \quad (14)$$

where the variable X has K possible values and s_k is the probability that X is equal to x_k. To obtain the probability that X meets some condition $X\in\theta$, the coefficients of the polynomial $U(z)$ should be summed for every term with $x_k\in\theta$. This can be done using the operator δ as follows:

$$Pr\{X \in \theta\} = \delta(U(z), \theta) = \sum_{k=1}^{K} s_k 1(x_k \in \theta). \quad (15)$$

In our case, the polynomial $U(z)$ can define system unit output distribution, i.e. it can represent all the possible states of the system by relating the probabilities of each state k to voting results corresponding to unit outputs $d_j(I)$ ($1 \le j \le n$) in that state. Each WVS state k can be characterized by two indices: state probability s_k and total score of unit votes at state k G_k.

Consider the UF $U(z) = \sum_{k=1}^{K} s_k z^{G_k}$, which relates these two indices. Using this polynomial one can describe output distributions of a system consisting of an individual unit j as

$$u_I^j(z) = \sum_{k=1}^{3} s_{jk} z^{G_{jk}} \quad (16)$$

where for I=1

$$\begin{aligned} &s_{j1} = q_{10}^{(j)}, \quad G_{j1} = -\tau w_j^0, \\ &s_{j2} = q_{1x}^{(j)}, \quad G_{j2} = 0, \\ &s_{j3} = q_{11}^{(j)} = (1 - q_{10}^{(j)} - q_{1x}^{(j)}), \quad G_{j3} = (1-\tau) w_j^1. \end{aligned} \quad (17)$$

and for I=0

$$\begin{aligned} &s_{j1} = q_{01}^{(j)}, \quad G_{j1} = (1-\tau) w_j^1, \\ &s_{j2} = q_{0x}^{(j)}, \quad G_{j2} = 0, \\ &s_{j3} = q_{00}^{(j)} = (1 - q_{01}^{(j)} - q_{0x}^{(j)}), \quad G_{j3} = -\tau w_j^0. \end{aligned} \quad (18)$$

The first state corresponds to a wrong decision, the second state corresponds to abstention and the third state corresponds to the right decision. To obtain the UF of a system containing first two voting units with their individual UF $u_I^1(z)$ and $u_I^2(z)$ the following composition operator can be used:

$$U_I^2(z) = \Omega(u_I^1(z), u_I{}^2(z)) = \Omega\ (\sum_{i=1}^{3} s_{1i} z^{G_{1i}}, \sum_{k=1}^{3} s_{2k} z^{G_{2k}}) = \sum_{i=1}^{3} \sum_{k=1}^{3} s_{1i} s_{2k} z^{G_{1i} + G_{2k}}. \quad (19)$$

One can see that the resulting UF $U_I^2(z)$ represents all the possible states of the two-unit WVS. Indeed, by multiplying corresponding probabilities and by summing scores one obtains polynomial

with 3^2=9 terms corresponding to all the possible states of the two-unit WVS. Each term describes individual WVS state by relating its probability and the total system score.

Given the UF for first *m*-1 voting units $U_I^{m-1}(z)$, one can easily obtain the UF for first *m* voting units $U_I^m(z)$ as:

$$U_I^m(z) = \Omega(U_I^{m-1}(z), u_I^m(z)) = \sum_{k=1}^{K_m} s_k^m z^{G_k^m} . \quad (20)$$

Consecutively applying the equation (20) for *m* = 2, …, *n* one can obtain the UF for the entire WVS $U_I^n(z)$. Note that while the total number of different states is 3^n, many of these states can result in the same values of score *G*. Therefore the total number of terms K_n in $U_I^n(z)$ can be much less then 3^n because of like terms collection. In the worst case the complexity of the procedure that obtains the UF $U_I^n(z)$ is O(3^n).

Using the δ operator (15) over $U_I^m(z)$ one can obtain probabilities that the proposition *I* will be accepted or rejected after voting of first *m* units as

$$h_{I0}^m = \delta(U_I^m(z),\ G_k^m < (\tau-1)V_{m+1}^1) = \sum_{k=1}^{K_m} s_k^m 1(G_k^m < (\tau-1)V_{m+1}^1), \quad (21)$$

(which corresponds to the sum of probabilities of scores meeting condition (12) of proposition rejection) and

$$h_{I1}^m = \delta(U_I^m(z),\ G_k^m \geq \tau V_{m+1}^0) = \sum_{k=1}^{K_m} s_k^m 1(G_k^m \geq \tau V_{m+1}^0). \quad (22)$$

(which corresponds to the sum of probabilities of scores meeting condition (13) that proposition cannot be rejected).

Terms meeting the conditions $G_k^m < (\tau-1)V_{m+1}^1$ or $G_k^m \geq \tau V_{m+1}^0$ correspond to the scores that guarantee that the system can make decisions independently of decisions of voting units *m*+1, …, *n*. Therefore, these terms can be removed from the UF $U_I^m(z)$ without loosing any information when the next UF $U_I^{m+1}(z)$ is determined.

The sum $h_{I0}^m + h_{I1}^m$ determines the probability that the WVS decision is made at time t_m.

Summing the probabilities h_{I0}^m and h_{I1}^m for m = 1, …, n we can obtain the overall probabilities of WVS decisions: $\sum_{m=1}^{n} h_{I0}^m$ gives the overall probability that the proposition I is rejected: Q_{I0}, $\sum_{m=1}^{n} h_{I1}^m$ gives the overall probability that the proposition I is not rejected: $Q_{I1}+Q_{Ix}$. Observe that when $m = n$, $V_{m+1}^0 = 0$ and $h_{I1}^n = \delta(U_I^n(z),\ G_k^n \geq 0)$ gives the sum of probabilities that the WVS accepts the proposition and that it abstains Q_{Ix}). Therefore the probability of proposition acceptance Q_{11} given I=1 is obtained as follows

$$Q_{11}= \sum_{m=1}^{n} h_{11}^m - Q_{1x}= \sum_{m=1}^{n} h_{11}^m - \prod_{j=1}^{n} q_{1x}^{(j)}. \quad (23)$$

Since events I=0 and I=1 are mutually exclusive, the entire WVS reliability Pr{$D(I)$=I} can be defined as Pr{$D(I)$=0 | I=0} Pr{I=0}+Pr{$D(I)$=1 | I=1} Pr{I=1} and calculated as follows:

$$R= p_0Q_{00}+p_1Q_{11}. \quad (24)$$

Having the times of units' voting and the probabilities that the WVS decision is made after voting of m units one can obtain the expected decision time as

$$T = p_0 \sum_{m=1}^{n}(h_{00}^m + h_{01}^m)t_m + p_1 \sum_{m=1}^{n}(h_{10}^m + h_{11}^m)t_m . \quad (25)$$

3.2. Algorithm for evaluating WVS reliability and expected decision time

Using the technique described in the previous section one can obtain the system reliability and the expected decision time by the following procedure:

1. Assign $U_0^0(z)=U_1^0(z)=1$; $Q_{00}=Q_{11}=T=0$.
2. For each voting unit j define two UF $u_0^j(z)$ and $u_1^j(z)$ in the form (16) using Eq. (17) and (18).
3. For m=1,…,n+1 obtain $(\tau-1)V_m^1$ and τV_m^0 using Eq. (9).
4. For m=1,…,n
   4.1. Obtain $U_0^m(z)$ and $U_1^m(z)$ using Eq. 20.
   4.2. Obtain h_{00}^m, h_{01}^m from $U_0^m(z)$ and h_{10}^m, h_{11}^m from $U_1^m(z)$ using Eq. (21) and (22).

4.3. Add h_{00}^m to Q_{00}, add h_{11}^m to Q_{11}.

4.4. Add $p_0 t_m(h_{00}^m + h_{01}^m) + p_1 t_m(h_{10}^m + h_{11}^m)$ to T.

4.5. Remove the terms meeting conditions (12) and (13) from $U_0^m(z)$ and $U_1^m(z)$.

5. Subtract $\prod_{j=1}^{n} q_{1x}^{(j)}$ from Q_{11}.

6. Calculate WVS reliability R using Eq. (24).

3.3. Example of WVS reliability and expected decision time evaluation

Given

$n = 3,\ p_0 = p_1 = 0.5,\ \tau = 0.6,$

$q_{01}^{(1)} = 0.02,\quad q_{10}^{(1)} = 0.02,\quad q_{0x}^{(1)} = q_{1x}^{(1)} = 0.01,\quad w^0{}_1 = 3,\quad w^1{}_1 = 5,\quad t_1=1;$

$q_{01}^{(2)} = 0.02,\quad q_{10}^{(2)} = 0.05,\quad q_{0x}^{(2)} = q_{1x}^{(2)} = 0.02,\quad w^0{}_2 = 4,\quad w^1{}_2 = 3,\quad t_2=2;$

$q_{01}^{(3)} = 0.01,\quad q_{10}^{(3)} = 0.03,\quad q_{0x}^{(3)} = q_{1x}^{(3)} = 0.0,\quad w^0{}_3 = 3,\quad w^1{}_3 = 2,\quad t_3=4.$

Following step 1 of the algorithm, assign:

$U_0^0(z)=U_1^0(z)=1;\ Q_{00}=Q_{11}=T=0.$

Following step 2 of the algorithm, obtain:

$(1-\tau)w^1{}_1 = 2,\ (1-\tau)w^1{}_2 = 1.2,\ (1-\tau)w^1{}_3 = 0.8;$

$-\tau w^0{}_1 = -1.8,\ -\tau w^0{}_2 = -2.4,\ -\tau w^0{}_3 = -1.8\ .$

The UF for the individual voting units are:

$u_0^1(z) = 10^{-2}(2z^2+z^0+97z^{-1.8}),\qquad u_1^1(z) = 10^{-2}(2z^{-1.8}+z^0+97z^2);$

$u_0^2(z) = 10^{-2}(2z^{1.2}+2z^0+96z^{-2.4}),\quad u_1^2(z) = 10^{-2}(5z^{-2.4}+2z^0+93z^{1.2});$

$u_0^3(z) = 10^{-2}(1z^{0.8}+99z^{-1.8}),\qquad u_1^3(z) = 10^{-2}(3z^{-1.8}+97z^{0.8}).$

Following step 3 of the algorithm, obtain:

$\tau V^0{}_2 = \tau(w^0{}_2 + w^0{}_3) = 0.6\cdot 7 = 4.2,\ \tau V^0{}_3 = \tau w^0{}_3 = 0.6\cdot 3 = 1.8,\ \tau V^0{}_4 = 0;$

$(\tau-1)V^1{}_2 = (\tau-1)(w^1{}_2 + w^1{}_3) = -0.4\cdot 8 = -3.2,$

$(\tau-1)V^1{}_3 = (\tau-1)w^1{}_3 = -0.4\cdot 2 = -0.8,\ (\tau-1)V^1{}_3 = 0;$

Following step 4 of the algorithm, obtain:

$U_0^1(z)=u_0^1(z)$, $U_1^1(z)=u_1^1(z)$.

The UF $U_0^1(z)$ and $U_1^1(z)$ do not contain terms that meet conditions (12) and (13). This means that the WVS cannot make any decision based on voting of the first unit and

$h_{00}^1 = h_{01}^1 = h_{10}^1 = h_{11}^1 = 0$.

UF for subsystem consisting of two units are

$U_0^2(z) = \Omega(u_0^1(z), u_0^2(z))=$

$10^{-4}(\mathbf{4z^{3.2}}+\mathbf{4z^{2}}+192z^{-0.4}+2z^{1.2}+2z^{0}+\boldsymbol{96z^{-2.4}}+194z^{-0.6}+\boldsymbol{194z^{-1.8}}+\boldsymbol{9312z^{-4.2}})$,

$U_1^2(z) = \Omega(u_1^1(z), u_1^2(z))=$

$10^{-4}(\boldsymbol{10z^{-4.2}}+\boldsymbol{4z^{-1.8}}+186z^{-0.6}+\boldsymbol{5z^{-2.4}}+2z^{0}+93z^{1.2}+485z^{-0.4}+\mathbf{194z^{2}}+\mathbf{9021z^{3.2}})$.

The terms meeting condition (12) are marked in bold italic, the terms meeting condition (13) are marked in bold. According to steps 4.2 and 4.3 of the algorithm:

$h_{00}^2 + h_{01}^2 = 10^{-4}(4+4+96+194+9312) = 0.\ 961$; $Q_{00} = h_{00}^2 = 10^{-4}(96+194+9312) = 0.9602$,

$h_{10}^2 + h_{11}^2 = 10^{-4}(10+4+5+194+9021) = 0.9234$; $Q_{11} = h_{11}^2 = 10^{-4}(194+9021) = 0.9215$.

After removing the marked terms the UF take the form

$U_0^2(z) = 10^{-4}(192z^{-0.4}+2z^{1.2}+2z^{0}+194z^{-0.6})$,

$U_1^2(z) = 10^{-4}(186z^{-0.6}+2z^{0}+93z^{1.2}+485z^{-0.4})$.

UF for subsystem consisting of three units are

$U_0^3(z) = \Omega(U_0^2(z), u_0^3(z)) = 10^{-6}(192z^{-0.4}+2z^{1.2}+2z^{0}+194z^{-0.6})(1z^{0.8}+99z^{-1.8})$

$= 10^{-6}(\mathbf{192z^{0.4}}+\mathbf{2z^{2.0}}+\mathbf{2z^{0.8}}+\mathbf{194z^{0.2}}+\boldsymbol{19008z^{-2.2}}+\boldsymbol{198z^{-0.6}}+\boldsymbol{198z^{-1.8}}+\boldsymbol{19206z^{-2.4}})$,

$U_1^3(z) = \Omega(U_1^2(z), u_1^3(z)) = 10^{-6}(186z^{-0.6}+2z^{0}+93z^{1.2}+485z^{-0.4})(3z^{-1.8}+97z^{0.8})$

$= 10^{-6}(\boldsymbol{558z^{-2.4}}+\boldsymbol{6z^{-1.8}}+\boldsymbol{279z^{-0.6}}+\boldsymbol{1455z^{-2.2}}+\mathbf{18042z^{0.2}}+\mathbf{194z^{0.8}}+\mathbf{9021z^{3.0}}+\mathbf{47045z^{0.4}})$.

In the final UF all of the terms meet either condition (12) or (13). The terms meeting condition (12) are marked in bold italic, the terms meeting condition (13) are marked in bold.

$h_{00}^3 + h_{01}^3 = 10^{-6}(192+2+2+194+19008+198+198+19206) = 0.039$;

$Q_{00} = 0.9602 + h_{00}^3 = 0.9602 + 10^{-6}(19008+198+198+19206) = 0.99881;$

$h_{10}^3 + h_{11}^3 = 10^{-6}(558+6+279+1455+18042+194+9021+47045) = 0.0766;$

$Q_{11} = 0.9215 + h_{11}^3 = 0.9215 + 10^{-6}(18042+194+9021+47045) = 0.995802.$

Since $Q_{1x} = q_{1x}^{(1)} q_{1x}^{(2)} q_{1x}^{(3)} = 0$, $Q_{11} - Q_{1x} = Q_{11} = 0.995802$.

The WVS reliability is

$R = p_0 Q_{00} + p_1 Q_{11} = 0.5 \cdot 0.99881 + 0.5 \cdot 0.995802 = 0.997306$.

The expected decision time is

$T = 0.5(0.961 \cdot 2 + 0.039 \cdot 4) + 0.5(0.9234 \cdot 2 + 0.0766 \cdot 4) = 2.1156$.

4. Solving the optimization problem

4.1. Optimization procedure

In order to solve the optimization problem (2) the same procedure is used that has been applied in [9] for solving the optimization problem (1). This procedure uses a genetic algorithm, which is based on the principle of evolutionary search in solution space.

Basic notions of GA are originally inspired by biological genetics. GA operates with "chromosomal" representation of solutions, where crossover, mutation and selection procedures are applied. This representation requires the solution to be coded as a finite length string. The natural representation of a WVS weight distribution is by an $2n+1$-length integer string in which the values in $2j-1$ and $2j$ position corresponds to the weights w^0_j and w^1_j of j-th unit of the WVS and the value in position $n+1$ corresponds to the threshold. The unit weights are further normalized in such a way that their total weight is always equal to some constant.

The detailed information on GA can be found in Goldberg's comprehensive book [16], and recent developments in GA theory and practice can be found in books [17, 18]. The basic structure of the version of GA, referred to as GENITOR [19], is as follows:

First, an initial population of N_S randomly constructed solutions (strings) is generated. Within this population, new solutions are obtained during the genetic cycle by using crossover and mutation operators. The crossover produces a new solution (offspring) from a randomly selected pair of parent solutions, facilitating the inheritance of some basic properties from the parents by the offspring. The probability of selecting the solution as a parent is proportional to the rank of this solution. (All the solutions in the population are ranked in order of their fitness increase). In this work, we use the so-called two point (or fragment) crossover operator which creates the offspring string for the given pair of parent strings by copying string elements belonging to the fragment between two randomly chosen positions from the first parent and by copying the rest of string elements from the second parent.

Each offspring solution undergoes mutation, which results in slight changes to the offspring's structure and maintains a diversity of solutions. This procedure avoids premature convergence to a local optimum and facilitates jumps in the solution space. The positive changes in the solution code, created by the mutation can be later propagated throughout the population via crossovers. In our GA, the mutation procedure swaps elements initially located in two randomly chosen positions on the string. Example of crossover and mutation procedures can be found in [8].

Each new solution is decoded and its objective function (fitness) value is estimated. In order to find the solution of Eq. (2) the fitness function is defined as:

$$F = R - \alpha \cdot \min(T-T^*,0), \qquad (26)$$

where α is a penalty coefficient. For solutions with $T<T^*$ the fitness of the solution depends only on WVS reliability. The solution fitness is used to compare different solutions.

The comparison is accomplished by a selection procedure that determines which solution is better: the newly obtained solution or the worst solution in the population. The better solution joins the population, while the other is discarded. If the population contains equivalent solutions following selection, redundancies are eliminated and the population size decreases as a result. After

new solutions are produced N_{rep} times, new randomly constructed solutions are generated to replenish the shrunken population, and a new genetic cycle begins.

The GA is terminated after N_c genetic cycles. The final population contains the best solution achieved.

4.2. Illustrative example

A WVS consists of six voting units with voting times and fault probabilities presented in Table 1. The optimal voting unit weights and thresholds and the parameters of the optimal WVS obtained for T^*=35 (when p_0=0.7, p_0=0.5, p_0=0.3) are presented in Tables 2 and 3. The system abstention probabilities do not depend of its weights and threshold. For any solution $Q_{0x}=0.868\times10^{-7}$, $Q_{1x}=1.89\times10^{-7}$.

It can be seen that for $p_0\neq0.5$ the WVS takes advantage of knowledge about statistical asymmetry of the input and provides greater reliability than in the case when p_0=0.5. Observe that when $p_0>0.5$ the WVS provides Q_{00} greater than Q_{11} and, vice versa, when $p_0<0.5$ $Q_{00}<Q_{11}$.

Table 1. Parameters of voting units

No of unit j	t_j	$q_{01}^{(j)}$	$q_{0x}^{(j)}$	$q_{10}^{(j)}$	$q_{1x}^{(j)}$
1	10	0.22	0.31	0.29	0.12
2	12	0.35	0.07	0.103	0.30
3	38	0.24	0.08	0.22	0.15
4	48	0.10	0.05	0.2	0.01
5	55	0.08	0.10	0.15	0.07
6	70	0.08	0.01	0.10	0.05

Table 2. Optimal weights for T^*=35

	No of unit j	1	2	3	4	5	6
	w^1_j	0.018	0.240	0.564	0.300	0.388	0.476
p_0=0.7	w^0_j	1.958	0.018	0.370	2.487	1.005	0.176
	w^1_j	0.017	2.367	0.497	0.017	0.635	1.475
p_0=0.5	w^0_j	2.281	0.360	0.189	1.561	0.566	0.034
	w^1_j	0.019	2.597	1.243	0.019	0.019	0.742
p_0=0.3	w^0_j	1.688	0.334	0.204	1.967	0.909	0.260

Table 3. Parameters of optimal WVS

	p_0=0.7	p_0=0.5	p_0=0.3
τ	0.76	0.50	0.45
Q_{00}	0.9798	0.9005	0.8477
Q_{01}	0.0202	0.0995	0.1523
Q_{10}	0.1611	0.0719	0.0283
Q_{11}	0.8389	0.9281	0.97166
R	0.9375	0.9143	0.9345
T	34.994	34.987	34.994

The R vs. T tradeoff curves for the WVS are presented in Fig. 2. These curves are obtained by solving the optimization problem (2) for different values of time constraint T^*.

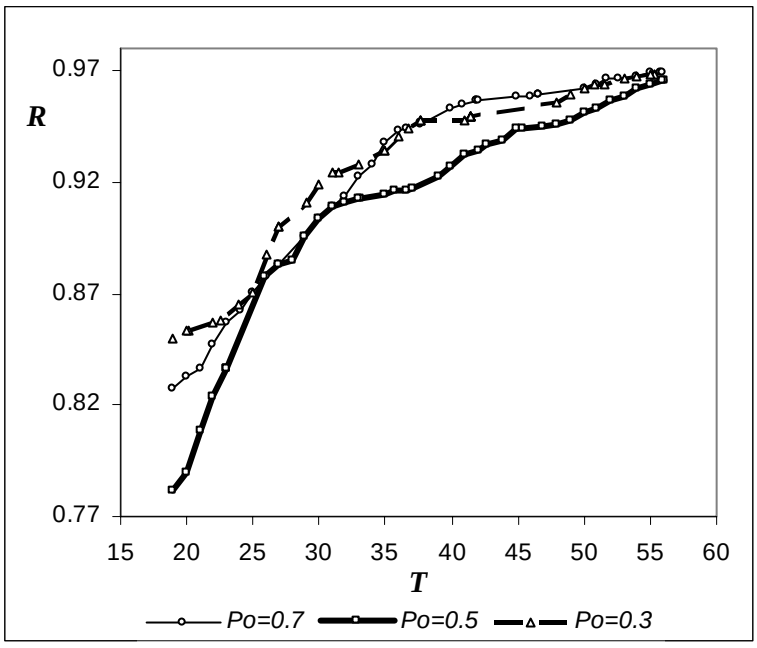


Fig. 2. Reliability vs. expected decision time for p_0=0.7, p_0=0.5, p_0=0.3.

4. Conclusions and further research

It is shown that if the weighted voting system consists of n units that need different time to produce their outputs, its decision time depends on the distribution of unit weights and on the value of the threshold and that a tradeoff exists between the system reliability and its rapidity.

An algorithm that finds the system parameters maximizing its reliability under constraint imposed on the expected system decision time is suggested. The algorithm is based on the universal generating function technique. The computational complexity of the algorithm is in the worst case $O(3^n)$.

Further research can be devoted to analysis of weighted voting systems composed from voting units with variable decision times, systems with dependent voting units, systems with weights adaptation etc.

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